REACHING ROW FIVE IN SOLITAIRE ARMY

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Abstract

Solitaire Army is a peg solitaire problem which involves trying to advance a peg as far as possible up the y-axis from an initial configuration consisting of an infinite half-plane of pegs. It is known that row four is reachable, but row five is not reachable in any finite number of moves. We present a well-defined model of the game permitting infinitely many moves, and a solution within that model which reaches row five. We also justify our choice of model, and generalise the infinite solution to n dimensions.

1. Introduction

Solitaire Army is a peg solitaire problem played on the infinite board $\mathbb{Z}^2$. The initial configuration consists of pegs at precisely $\{(x, y) : y \leqslant 0\}$. Acceptable moves are those of standard peg solitaire: a peg may jump exactly two units in any of the four orthogonal directions, provided the target point is empty and the point in between is full. The peg jumped over is removed. The aim is to advance a peg as far as possible in the positive y-direction, or equivalently to reach the specific point $(0, y)$ for y as large as possible.

It is proved in [1] that $(0, 5)$ is not reachable. The proof is as follows: to each point (x, y), we assign the value $\phi^{y-|x|}$, where ϕ is the usual golden ratio $\frac{1}{2}(1+\sqrt{5})$, and consider the total value of the board to be the sum of the values of all occupied points. Then it is easy to see that no individual move can ever increase the total board value: a move upwards, or inwards toward the y-axis, replaces pegs of value k and $k\phi$ (for some k) with one of value $k\phi^2$, which is of exactly equal value to the two original pegs put together, since $1 + \phi = \phi^2$. Hence such a move leaves the total board value unchanged. All other moves (outward, downward, or crossing the y-axis) decrease it.

The total value of the initial configuration can be computed as a sum of geometric progressions; it comes to $'^{5}$, which is exactly equal to the value of a single peg at $(0; 5)$. Hence, in order to reach the fifth row one would have to use up every single peg in the initial configuration. This would require an infinite number of moves, and hence cannot be done.

However, enough value is present on the board *in principle* to reach $(0; 5)$, and only the restriction to a finite subset of the starting pieces prevents us from accessing all of that value. So it is natural to wonder if we could reach $(0; 5)$ in the absence of that restriction. And the answer is yes: in this paper we shall present a small relaxation of the rules to permit infinitely many moves, and then demonstrate a solution which reaches the fifth row.

2. Ground rules

To begin with, we must define exactly what we mean by the idea of making infinitely many moves in a peg solitaire game.

We will frequently want to make an infinite sequence of moves, and then make further moves after it has "finished". One way to do this might be to consider a transfinite sequence of moves indexed by ordinals. However, this turns out not to work (see section 6 for a proof).

Another natural way to formalise this is to consider making our moves at instants of real-valued time. For example, we could make a sequence of moves at every instant in the set $\{1 - 2^{-k} : \mathbf{k} \in \mathbb{N}\}$, and at $\mathbf{t} = 1$ we would have completed our entire sequence and could proceed to make more moves.

We must also impose a constraint on changing the status of a point infinitely often. If we were to make an infinite sequence of moves as described above, and an infinite subset of those moves emptied or filled the point $(0; 0)$, then it would be difficult to justify describing that point as either empty or full at the end of the sequence. If we restrict ourselves to changing any point at most finitely often, however, this problem cannot arise.

An obvious constraint would be to rule that every point must be involved in finitely many moves. However, it turns out that this is not ideal: we will instead impose the stronger condition that all points must be involved in a *bounded* number of moves, with a bound that applies uniformly across the whole of $\mathbb{Z}^2$. In section 4 we discuss the reason for this.

So, we could present a completely formal definition of our problem as follows:

The changing state of the board is represented by a function $\mathbf{f} : \mathbb{Z}^2 \times \mathbb{R} \to \{0; \frac{1}{2}; 1\}$, such that $\mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t})$ gives the state of the point $(\mathbf{x}; \mathbf{y})$ at time $\mathbf{t}$. The value 0 means the point is empty; 1 means it is full; $\frac{1}{2}$ means that a move affecting that point is taking place at that precise instant.

Our function $\mathbf{f}$ must satisfy a number of constraints:

1. **All changes of state occur by means of a move.** Given $\mathbf{x}; \mathbf{y} \in \mathbb{Z}$ and $\mathbf{t}; \mathbf{u} \in \mathbb{R}$ with $\mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t}) = 0$ and $\mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{u}) = 1$, there must exist some $\mathbf{v} \in (\mathbf{t}; \mathbf{u})$ with $\mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{v}) = \frac{1}{2}$.
2. **Only one move at a time.** Given any fixed $\mathbf{t}$, the set $\{(\mathbf{x}; \mathbf{y}) \in \mathbb{Z}^2 : \mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t}) = \frac{1}{2}\}$ has size either 0 or 3.
3. **Each point is involved in a bounded number of moves.** There exists $\mathbf{N} \in \mathbb{N}$ such that, for any $\mathbf{x}; \mathbf{y} \in \mathbb{Z}$, the set $\{\mathbf{t} \in \mathbb{R} : \mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t}) = \frac{1}{2}\}$ has size at most $\mathbf{N}$.

As a consequence of rule 3, it is meaningful to talk about the state of a point "before" or "after" a given move, since for fixed $\mathbf{x}; \mathbf{y}; \mathbf{t}$, there exists $\quad > 0$ such that $\mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t} - \quad)$ and $\mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t} + \quad)$ are each constant for all $\quad \in (0; \quad)$. This allows us to state our next rule in a well-defined manner:

4. **Legal peg solitaire moves only.** Given any time $\mathbf{t}$ at which a move takes place, in other words one for which the set $\mathbf{S} = \{(\mathbf{x}; \mathbf{y}) \in \mathbb{Z}^2 : \mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t}) = \frac{1}{2}\}$ has size 3, $\mathbf{S}$ must consist of three points which are either of the form $\{(\mathbf{x} - 1; \mathbf{y}); (\mathbf{x}; \mathbf{y}); (\mathbf{x} + 1; \mathbf{y})\}$ or $\{(\mathbf{x}; \mathbf{y} - 1); (\mathbf{x}; \mathbf{y}); (\mathbf{x}; \mathbf{y} + 1)\}$. Just before the move, the states of the outer two points must be different, and the state of the central point must be 1. After the move, all three points must have changed state.

We would like to say that any function $\mathbf{f}$ satisfying all of the above constraints is deemed to be a legal sequence of play in our revised Solitaire Army game.

We then seek such an $\mathbf{f}$ which also satisfies our game-winning conditions:

5. **Starting position.** There exists some $\mathbf{t}_{\text{start}}$ such that $\mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t}_{\text{start}})$ is equal to 1 if $\mathbf{y} \leqslant 0$, and to 0 otherwise.
6. **Winning position.** There exists some $\mathbf{t}_{\text{end}} > \mathbf{t}_{\text{start}}$ such that $\mathbf{f}(0; 5; \mathbf{t}_{\text{end}}) = 1$.

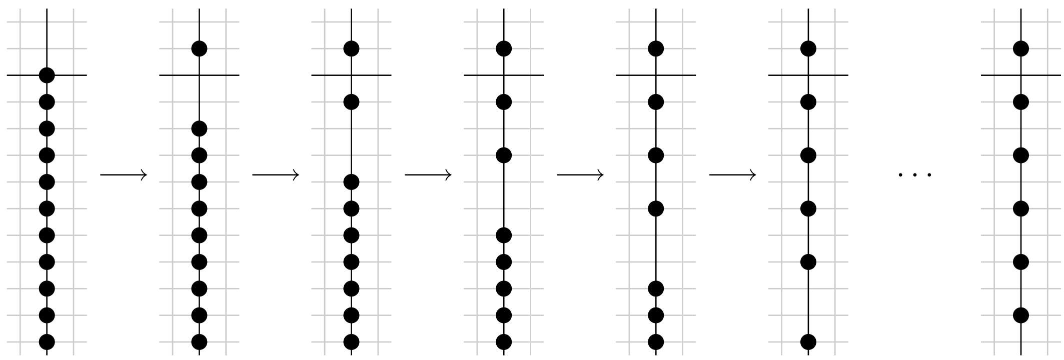
Now that we have defined this formal model of the game, we will revert in the following sections to informal descriptions of move sequences. The reader should find no difficulty in constructing a function $\mathbf{f}$ to match any of our descriptions.

3. The solution

3.1 Whoosh

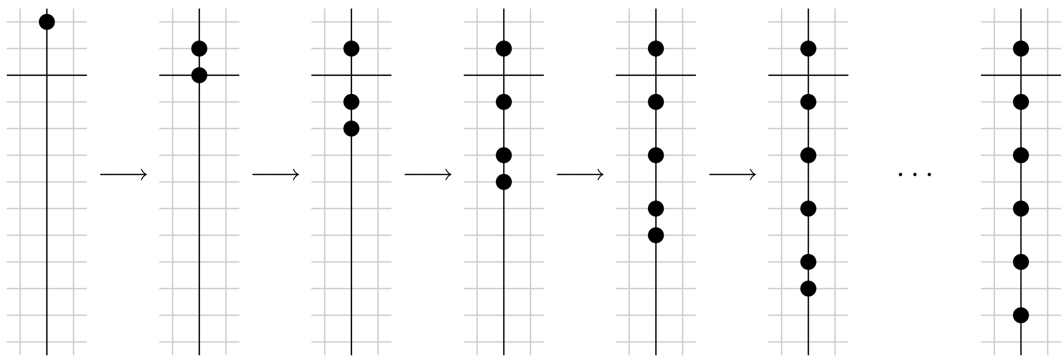
Our first example of a useful infinite move sequence converts an infinite set of pegs into a single peg with the same value as the original set. Specifically, we will convert an entire column of pegs at $\{(0; \mathbf{y}) : \mathbf{y} \leqslant 0\}$ into a single peg at $(0; 2)$.

We begin by performing an infinite sequence of upward jumps. We jump $(0; -1)$ upwards over $(0; 0)$, then we jump $(0; -3)$ upwards over $(0; -2)$, then $(0; -5)$ over $(0; -4)$, and so on.



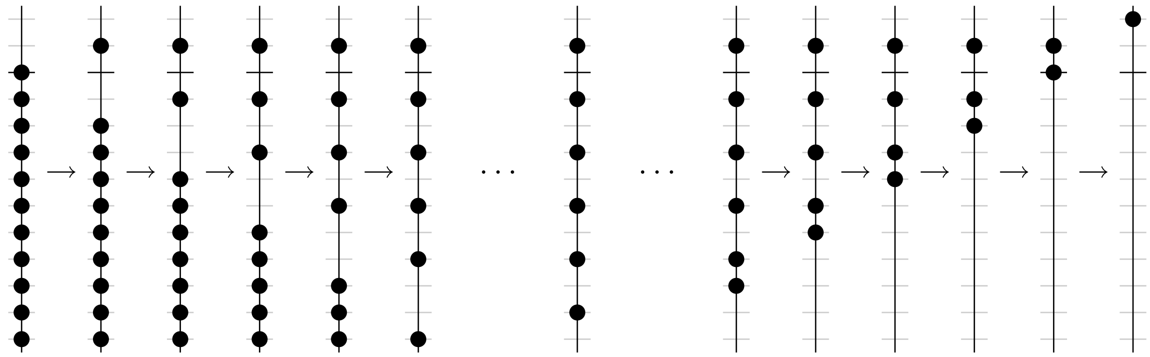
If we perform these moves at times (say) $0; \frac{1}{2}; \frac{2}{3}; \frac{3}{4}; \frac{4}{5}; :::$, then at time 1 the column will be in a state where the set of pegs present is exactly $\{(0; 1 - 2\mathbf{k}) : \mathbf{k} \geqslant 0\}$.

Now let us consider the other end of the sequence. Our initial column had board value '^{2}, so a single peg with the same value would have to be at $(0; 2)$. So let us start with a peg there, and make *backwards* peg solitaire moves (in which a peg is jumped over an empty point, filling it in). We jump $(0; 2)$ downwards to $(0; 0)$, and then jump that downwards to $(0; -2)$, and so on:



In order for this sequence to make sense when run in the normal direction, we must perform the moves at a sequence of time instants tending *downwards* towards a limit, rather than upwards as in the previous example. So we might, for example, perform the final move (leaving us with a single peg at $(0, 2)$) at time 2, the move before that at time $\frac{3}{2}$, the one before that at time $\frac{4}{3}$, then $\frac{5}{4}, \frac{6}{5}, \frac{7}{6}$, and so on.

The final configuration after performing this sequence of reverse moves is exactly the same as the final configuration after the first sequence of forward moves. So, if we perform the moves in the first sequence at times $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$, and then the moves in the second sequence at times $\ldots, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, 2$, then in bounded time we have converted our original column of pegs at $\{(0, y) : y \leqslant 0\}$ into a single peg at $(0, 2)$ with the same total value.



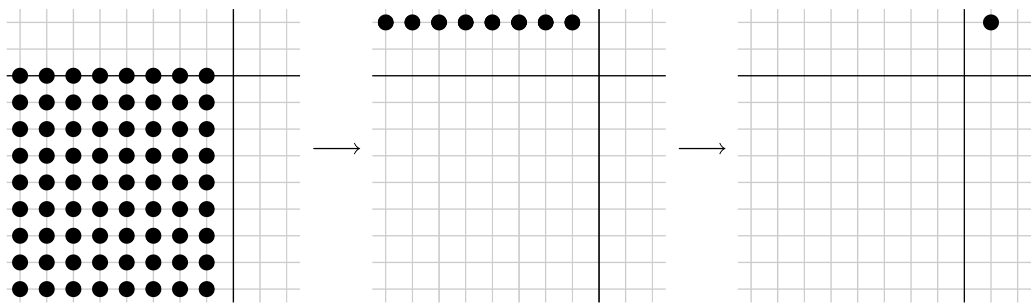
Of course, this sequence can be applied to any semi-infinite line of pegs in any direction, and will turn it into a single peg two spaces beyond its original end point.

We will use this basic move a lot; let us refer to it as a "whoosh".

3.2 Megawhoosh

We've presented a sequence of moves which converts a semi-infinite line of pegs into a single peg. We will now do the same thing in two dimensions: we want a "megawhoosh" which converts a quarter-plane, let us say $\{(x, y) : x < 0, y \leqslant 0\}$, into a single peg.

The most obvious way to do this is to take our quarter-plane, whoosh every line of it upwards, and then whoosh the resulting horizontal line sideways:



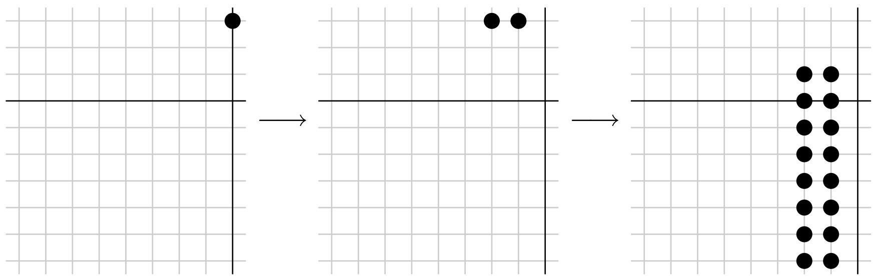
Unfortunately, this simple construction is off by one space from where we need to be. The final sideways whoosh has crossed the y-axis, and hence has decreased the board value. We would have liked our final peg to end up at $(0, 3)$, not $(1, 2)$.

We could move this entire procedure one space to the left, so that it transforms the quarter-plane $\{(\mathbf{x};\mathbf{y}) : \mathbf{x} < -1; \mathbf{y} \leqslant 0\}$ into a peg at $(0;2)$, in order to avoid decreasing board value. However, this would leave us with a column of pegs at $\{(-1;\mathbf{y}) : \mathbf{y} \leqslant 0\}$, which are now useless, because there's no way to bring them in to the central column (since moving them inwards would cross the $\mathbf{y}$-axis, and nothing remains further out to jump over them).

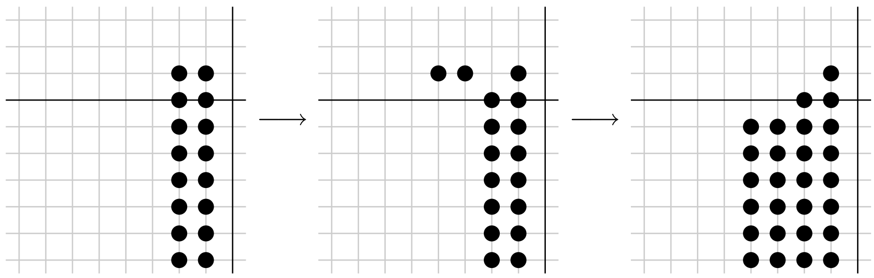
So, in fact, this simple approach doesn't work, and we need to do something completely different and somewhat more complex.

It turns out to be easier to describe our megawhoosh construction in reverse, as it was for the second half of the whoosh.

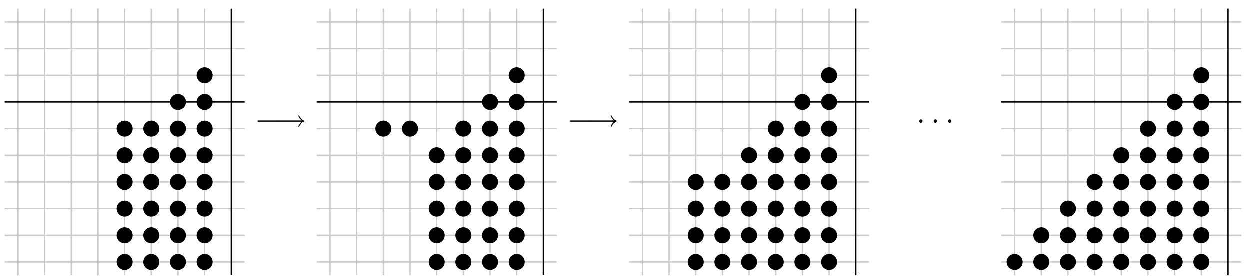
So we start with a single peg at $(0;3)$. Our first steps are to move it left, and then to whoosh both the resulting pegs downwards. (Viewed in reversed time, of course, a whoosh turns a single peg into a semi-infinite line.)



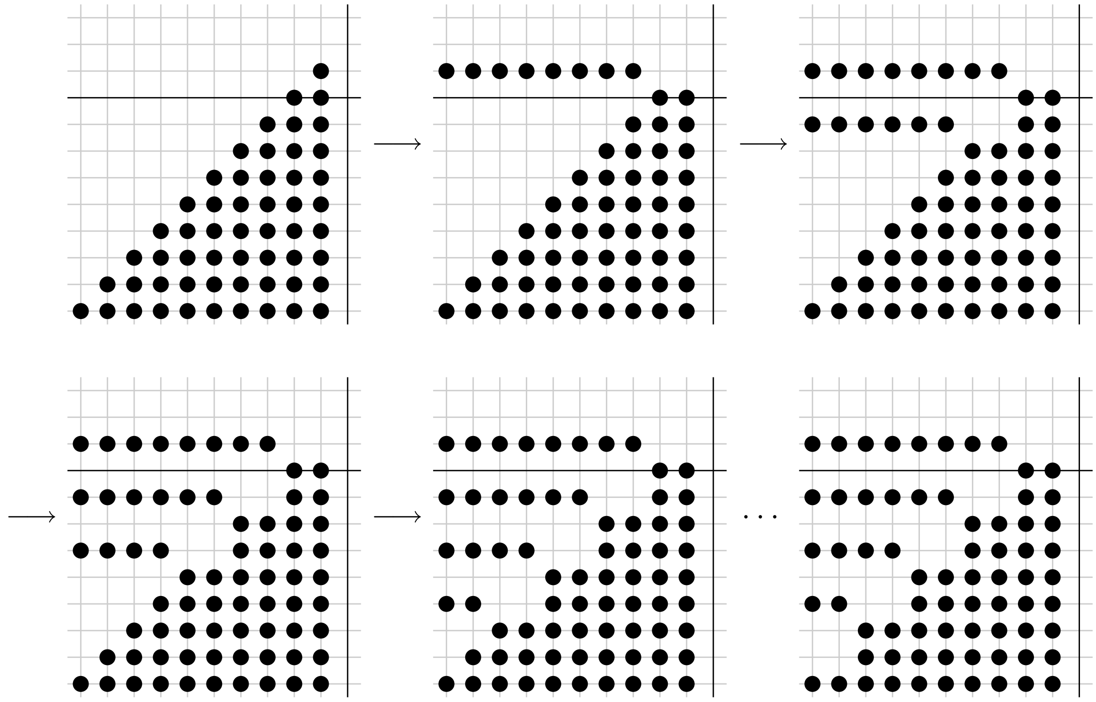
Now we do the same two steps starting with the topmost peg of the left-hand column:



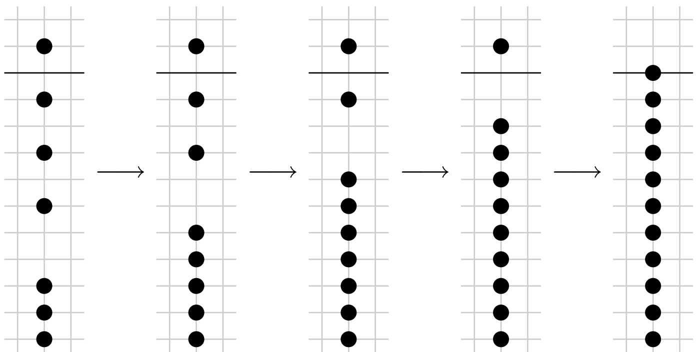
Then we do it again with the topmost peg of the leftmost column resulting from *that*, and continue in this way until we have generated an infinite staircase filling about half of our target quarter-plane:



Now we do a leftward whoosh with *every other* peg on the outer edge of that staircase, starting with the one at $(-1;1)$:



This leaves only a finite number of downward moves to be made in each column: we move the lowest "loose" piece downwards, then the next lowest, and so on until we have moved the piece at $(\mathbf{x};1)$ downwards to fill $(\mathbf{x};0)$ and $(\mathbf{x};-1)$. For example, in the column $\mathbf{x}=-9$:



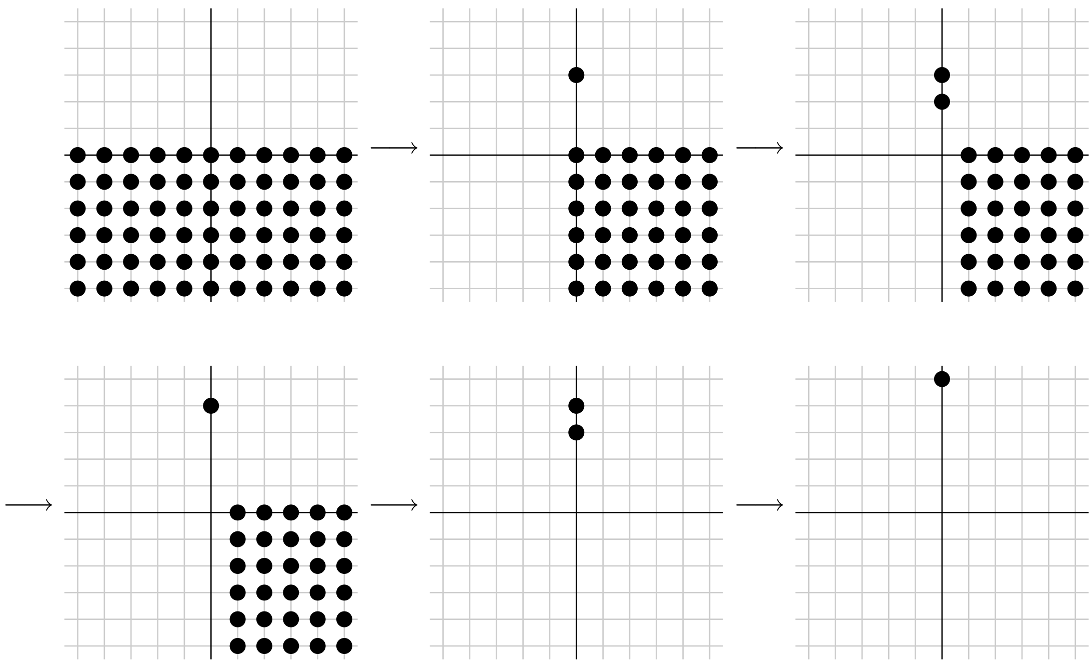
Since only finitely many downward moves are required in each column, we can enumerate all the downward moves required in the quarter-plane, and make them in a finite amount of time in the usual way. We have then successfully transformed a peg at $(0;3)$ into the entire quarter-plane $\{(\mathbf{x};\mathbf{y}) : \mathbf{x} < 0; \mathbf{y} \leqslant 0\}$. If we reverse this entire sequence, we end up with our desired "megawhoosh" move.

(In the above presentation we have assumed the ability to perform an infinite sequence of steps with each step itself including a doubly-infinite whoosh move. This does not present a problem in terms of the formal model given earlier: we first divide a finite length of time into infinitely many intervals, for example one of length 2^{-n} for each $n > 0$, and then each of those intervals can be infinitely subdivided as required for the whooshes and other moves that must be performed within it.)

3.3 Putting it all together

Having defined the whoosh and the megawhoosh, we can now quickly present the full solution.

1. Megawhoosh the quarter-plane $\{(x; y) : x < 0; y \leqslant 0\}$ up to a single peg at $(0; 3)$.
2. Whoosh the central column $\{(0; y) : y \leqslant 0\}$ up to a single peg at $(0; 2)$.
3. Jump this peg over the one at $(0; 3)$, reaching $(0; 4)$.
4. Perform the megawhoosh in mirror image about the y-axis, transforming the other quarter-plane $\{(x; y) : x > 0; y \leqslant 0\}$ into a single peg at $(0; 3)$.
5. Jump this peg over the one at $(0; 4)$, reaching $(0; 5)$.



We have successfully reached $(0; 5)$, and as predicted by the proof in [1] we have had to use every peg in the initial configuration to do it.

4. Consequences of removing the move bound

Rule 3 in section 2 imposes a uniform bound on the number of moves affecting any point, rather than the weaker and more obvious condition that each point must experience merely *finitely* many moves.

If we had used the weaker condition, some surprising sequences of play would have become valid. We now exhibit such a sequence, which generates a peg out of nothing at all.

As with the megawhoosh, this sequence is easier to describe in reverse. We begin with a peg at $(0;0)$, and we seek a sequence of moves $\mathbf{S}$ which destroys it completely within (say) the time interval $[0;1)$.

Let us move the peg upwards at time 0, so that it becomes two pegs at $(0;1)$ and $(0;2)$. Then we *recursively* perform the sequence $\mathbf{S}$ twice: we destroy the peg at $(0;2)$ by performing a copy of $\mathbf{S}$ translated upwards by 2 and compressed into the time interval $[\frac{1}{3};\frac{2}{3})$, and then we destroy the peg at $(0;1)$ by performing a second copy of $\mathbf{S}$ translated upwards by 1 and compressed into the time interval $[\frac{2}{3};1)$.

This is an informal description, which 'defines' $\mathbf{S}$ in terms of itself. We now formalise it by giving a definition of the actual function which gives the state of each point at any time instant. Since we are working in reverse solitaire, we will present this in its time-reversed form $\mathbf{g}(\mathbf{x};\mathbf{y};\mathbf{t})$, and define $\mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{t}) = \mathbf{g}(\mathbf{x};\mathbf{y};1-\mathbf{t})$ to be the forward function.

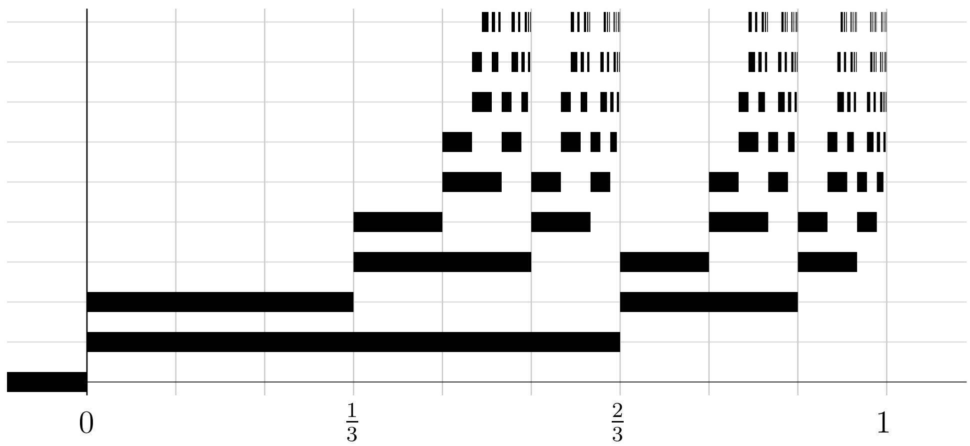
- For $\mathbf{t} \geqslant 1$, $\mathbf{g}(\mathbf{x};\mathbf{y};\mathbf{t}) = 0$ everywhere.
- For $\mathbf{t} < 0$, $\mathbf{g}(0;0;\mathbf{t}) = 1$, and $\mathbf{g}(\mathbf{x};\mathbf{y};\mathbf{t}) = 0$ everywhere else.
- If $\mathbf{x} \neq 0$ or $\mathbf{y} < 0$, then $\mathbf{g}(\mathbf{x};\mathbf{y};\mathbf{t}) = 0$ for all $\mathbf{t}$.
- For $\mathbf{t} \in [0;1)$, we determine the states of the points $\{(0;\mathbf{y}) : \mathbf{y} \geqslant 0\}$ by considering the infinite ternary expansion of $\mathbf{t}$. Construct a sequence by progressing through that ternary expansion as follows:
  - If we see the digit 1, we append the two terms $0;1$ to the sequence.
  - If we see the digit 2, we append the term 0 to the sequence.
  - If we see the digit 0, we append the three terms $\frac{1}{2};\frac{1}{2};\frac{1}{2}$ if the expansion terminates there (i.e. all subsequent digits are also zero), and otherwise we append the three terms $0;1;1$. In either case, we then stop, and all subsequent terms of the sequence are defined to be 0.

  The initial term of the resulting sequence gives the value of $\mathbf{g}(0;0;\mathbf{t})$; the next term gives $\mathbf{g}(0;1;\mathbf{t})$, and so on.

For example, here are some ternary expansions and their resulting sequences:

$$\begin{array}{lllllllllllll}
0{:}1212012{:}{:}{:} & \mapsto & 0; & 1; & 0; & 0; & 1; & 0; & 0; & 1; & 1; & 0; & \langle\text{repeat 0 forever}\rangle \\
0{:}1212 & \mapsto & 0; & 1; & 0; & 0; & 1; & 0; & \frac{1}{2}; & \frac{1}{2}; & \frac{1}{2}; & 0; & \langle\text{repeat 0 forever}\rangle \\
0{:}1111111{:}{:}{:} & \mapsto & 0; & 1; & 0; & 1; & 0; & 1; & 0; & 1; & 0; & 1; & \langle\text{repeat } 0;1 \text{ forever}\rangle
\end{array}$$

Here is a diagram illustrating the states of the points $(0;0)$ to $(0;9)$ during the (time-reversed) procedure:



This construction is entirely valid according to the rules, except that the number of moves affecting the point $(0;\mathbf{n})$ is unbounded: for $\mathbf{n} > 0$, it is equal to twice the $(\mathbf{n}+1)$th Fibonacci number. So $\mathbf{f}$ would represent a valid sequence of play if rule 3 were weakened to remove the uniform bound.

So if we were to remove the move bound, we would be able to reach the target point $(0;5)$ simply by creating a peg out of nothingness at $(0;5)$, without even touching the starting army. This does not really seem to be in the spirit of the original Solitaire Army problem.

5. Consequences of putting it back again

With the rules as presented in section 2, this sort of abuse is prevented: board value cannot be created from nothingness, similarly to the finite case. We now prove this.

Definition. A *value function* is a function $\mathbf{v} : \mathbb{Z}^2 \to [0;\infty)$, which allocates a value to every point, such that

$$\sum_{x,y\in\mathbb{Z}} \mathbf{v}(\mathbf{x};\mathbf{y}) < \infty$$

(in which the terms may be summed in any order, since this definition ensures that the sum is absolutely convergent).

Definition. Given a function $\mathbf{f}$ satisfying rules 1–4 in section 2, and a value function $\mathbf{v}$, we define the total board value at time $\mathbf{t}$ to be

$$\mathbf{V}(\mathbf{t}) = \sum_{x,y\in\mathbb{Z}} \mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{t})\,\mathbf{v}(\mathbf{x};\mathbf{y})$$

in which, similarly, the terms may be summed in any order.

Definition. Given $\mathbf{f}$ as above, we define

$$\mathbf{d}(\mathbf{x};\mathbf{y};\mathbf{t}) = \lim_{u\searrow t} \mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{u}) - \lim_{u\nearrow t} \mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{u})$$

or, in other words,

$$\mathbf{d}(\mathbf{x};\mathbf{y};\mathbf{t}) = \begin{cases} +1 & \text{if the move at time } \mathbf{t} \text{ creates a peg at } (\mathbf{x};\mathbf{y}) \\ -1 & \text{if the move at time } \mathbf{t} \text{ removes a peg at } (\mathbf{x};\mathbf{y}) \\ 0 & \text{if there is no move at time } \mathbf{t} \text{ which affects } (\mathbf{x};\mathbf{y}). \end{cases}$$

Given $\mathbf{f}$ and $\mathbf{v}$ as above, we then define the total change in board value at time $\mathbf{t}$ to be

$$\mathbf{D}(\mathbf{t}) = \sum_{x,y\in\mathbb{Z}} \mathbf{d}(\mathbf{x};\mathbf{y};\mathbf{t})\,\mathbf{v}(\mathbf{x};\mathbf{y})$$

in which there are always at most three non-zero terms in the sum.

Theorem 1. Let $\mathbf{f}$ be a function satisfying all of rules 1–4 in section 2, and let $\mathbf{v}$ be a value function. Let $\mathbf{t}_0;\mathbf{t}_1 \in \mathbb{R}$ be such that $\mathbf{t}_0 < \mathbf{t}_1$ and no move takes place at either $\mathbf{t}_0$ or $\mathbf{t}_1$. Then the sum

$$\sum_{\substack{t\in(t_0,t_1) \\ \exists (x,y)\in\mathbb{Z}^2 : f(x,y,t)=\frac{1}{2}}} \mathbf{D}(\mathbf{t})$$

(that is, the sum of $\mathbf{D}(\mathbf{t})$ over all move times in $(\mathbf{t}_0;\mathbf{t}_1)$) is well defined and is equal to $\mathbf{V}(\mathbf{t}_1) - \mathbf{V}(\mathbf{t}_0)$.

Proof. We can write the state $\mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{t}_1)$ of the point $(\mathbf{x};\mathbf{y})$ at time $\mathbf{t}_1$ in terms of its state at $\mathbf{t}_0$ and its changes of state between then and $\mathbf{t}_1$:

$$\mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{t}_1) = \mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{t}_0) + \sum_{t\in(t_0,t_1)} \mathbf{d}(\mathbf{x};\mathbf{y};\mathbf{t})$$

in which the sum has finitely many (in fact at most $\mathbf{N}$) non-zero terms.

The contribution to the total board value $\mathbf{V}(\mathbf{t}_1)$ from this point is therefore given by multiplying the above by $\mathbf{v}(\mathbf{x};\mathbf{y})$:

$$\mathbf{V}_{x,y}(\mathbf{t}_1) = \mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{t}_1)\,\mathbf{v}(\mathbf{x};\mathbf{y}) = \mathbf{f}(\mathbf{x};\mathbf{y};\mathbf{t}_0)\,\mathbf{v}(\mathbf{x};\mathbf{y}) + \sum_{t\in(t_0,t_1)} \mathbf{d}(\mathbf{x};\mathbf{y};\mathbf{t})\,\mathbf{v}(\mathbf{x};\mathbf{y}):$$

The total board value is therefore the sum of this expression over all $\mathbf{x;y}$:

$$\mathbf{V}(\mathbf{t}_1) = \sum_{x,y\in\mathbb{Z}} \left(\mathbf{f}(\mathbf{x;y;t}_0)\,\mathbf{v}(\mathbf{x;y}) + \sum_{t\in(t_0,t_1)} \mathbf{d}(\mathbf{x;y;t})\,\mathbf{v}(\mathbf{x;y}) \right)$$

in which each of the inner sums contains finitely many non-zero terms.

This series contains both positive and negative terms (since $\mathbf{d}(\mathbf{x;y;t})$ can be negative), but it is absolutely convergent: for each $\mathbf{x;y}$, there are at most $\mathbf{N}$ non-zero terms of the form $\mathbf{d}(\mathbf{x;y;t})\,\mathbf{v}(\mathbf{x;y})$, and one of the form $\mathbf{f}(\mathbf{x;y;t}_0)\,\mathbf{v}(\mathbf{x;y})$, so the sum of their absolute values is at most $(\mathbf{N}+1)\,\mathbf{v}(\mathbf{x;y})$. Hence the sum of the absolute values of the entire series is bounded above by

$$\sum_{x,y\in\mathbb{Z}} (\mathbf{N}+1)\,\mathbf{v}(\mathbf{x;y}) = (\mathbf{N}+1) \sum_{x,y\in\mathbb{Z}} \mathbf{v}(\mathbf{x;y})$$

which is finite, by definition of $\mathbf{v}$.

Since the series is absolutely convergent, we can reorder it. In particular, we can pull out the $\mathbf{f}(\mathbf{x;y;t}_0)\,\mathbf{v}(\mathbf{x;y})$ terms:

$$\begin{aligned}
\mathbf{V}(\mathbf{t}_1) &= \sum_{x,y\in\mathbb{Z}} \mathbf{f}(\mathbf{x;y;t}_0)\,\mathbf{v}(\mathbf{x;y}) + \sum_{x,y\in\mathbb{Z}} \sum_{t\in(t_0,t_1)} \mathbf{d}(\mathbf{x;y;t})\,\mathbf{v}(\mathbf{x;y}) \\
&= \mathbf{V}(\mathbf{t}_0) + \sum_{x,y\in\mathbb{Z}} \sum_{t\in(t_0,t_1)} \mathbf{d}(\mathbf{x;y;t})\,\mathbf{v}(\mathbf{x;y})
\end{aligned}$$

The remaining double sum contains three terms for every move between $\mathbf{t}_0$ and $\mathbf{t}_1$: one for each of the three points affected by that move. So we now enumerate the complete set of move times in that interval (of which there must be at most countably many – each of the countable set of points on the board is involved in finitely many moves) and reorder the series to bring those sets of three terms together:

$$\mathbf{V}(\mathbf{t}_1) = \mathbf{V}(\mathbf{t}_0) + \sum_{\substack{t\in(t_0,t_1) \\ \exists(x,y)\in\mathbb{Z}^2 : f(x,y,t)=\frac{1}{2}}} \sum_{x,y\in\mathbb{Z}} \mathbf{d}(\mathbf{x;y;t})\,\mathbf{v}(\mathbf{x;y})$$

in which the innermost sum over $\mathbf{x;y}$ always has exactly three terms.

That innermost sum is precisely the definition of $\mathbf{D}(\mathbf{t})$, giving us the result we are after:

$$\mathbf{V}(\mathbf{t}_1) = \mathbf{V}(\mathbf{t}_0) + \sum_{\substack{t\in(t_0,t_1) \\ \exists(x,y)\in\mathbb{Z}^2 : f(x,y,t)=\frac{1}{2}}} \mathbf{D}(\mathbf{t})$$

Since we have shown that the above sum is absolutely convergent, its value must be the same no matter in what order we enumerate the set of move times. □

Corollary 1. $\mathbf{V}(\mathbf{t})$ is a decreasing function iff $\mathbf{D}(\mathbf{t}) \leqslant 0$ for all $\mathbf{t}$. □

Corollary 2. $\mathbf{V}(\mathbf{t})$ is a constant function iff $\mathbf{D}(\mathbf{t}) = 0$ for all $\mathbf{t}$. □

Corollary 1 provides us with basically the same constraint on board value changes as in the finite game: for any value function of the form $\mathbf{v}(\mathbf{x};\mathbf{y}) = \varphi^{-(|x-X|+|y-Y|)}$ (for some constant $\mathbf{X};\mathbf{Y}$) there can exist no move with a positive $\mathbf{D}(\mathbf{t})$ and so board value must be a decreasing function of time. We are therefore assured that not only is the specific construction in section 4 ruled out, but *any* construction which manages to create a peg out of nothingness must be ruled out as well.

The board value definition in section 1 (used for the finite form of the problem) is not a valid value function according to the definition in this section, because it increases without bound as $\mathbf{y} \to \infty$. However, for any point $(\mathbf{X};\mathbf{Y})$ with $\mathbf{Y} > 5$, we can define $\mathbf{v}(\mathbf{x};\mathbf{y}) = \varphi^{-(|x-X|+|y-Y|)}$ as above, and this is sufficient to show that any such $(\mathbf{X};\mathbf{Y})$ is unreachable using the standard starting army $\{(\mathbf{x};\mathbf{y}) : \mathbf{y} \leqslant 0\}$. Hence, we can safely define the value function

$$\mathbf{v}(\mathbf{x};\mathbf{y}) = \begin{cases} \varphi^{y-|x|} & \text{if } \mathbf{y} \leqslant 5 \\ 0 & \text{if } \mathbf{y} > 5 \end{cases}$$

which matches the definition in section 1 for all reachable points, but which has a finite sum and hence is a permissible value function for the purposes of the above proof. In the following section we will use the term "board value" without qualification to mean this particular value function.

6. No well-ordered solution

The solution we presented in section 3 makes use of both forward infinite sequences of moves (with a definite beginning but no end) and backward infinite sequences (with an end but no beginning). The latter are somewhat counterintuitive: the second half of the whoosh move, in particular, feels as if it "shouldn't" work, because at the half-way point the column is completely composed of alternating pegs and spaces, and so it seems intuitively obvious that there should be no way to make any move at all starting from that position.

Backward infinite sequences are perfectly legal according to the rules in section 2, but it would certainly be even better if we could find a solution which did not require them. In other words, we would like a solution in which the moves are well-ordered. Failing that, the next best thing would be to have the moves well-ordered if the solution is viewed exactly in reverse; that way, we would have nothing *but* backward infinite sequences, but at least the solution would appear reasonably intuitive one way round.

In fact, within our rules, neither of these is possible. In this section we prove this.

Theorem 2. Given any function $\mathbf{f}$ representing a valid solution (i.e. satisfying all of rules 1–6 in section 2), there exists a strictly decreasing infinite sequence of time instants at which moves take place, and also a strictly increasing one. Hence the moves cannot be well-ordered in either the forward or backward direction.

Proof. Recall that the board value of a single peg at the target point $(0;5)$ is exactly equal to the value of the entire starting army. Therefore, Corollary 1 in section 5 implies that any valid solution must use up all the pegs in the initial configuration and end up with *only* one peg at $(0; 5)$; and Corollary 2 further implies that all moves made during a valid solution must have $\mathbf{D}(\mathbf{t}) = 0$, i.e. they are either upward moves or moves inward towards the $\mathbf{y}$-axis.

For $\mathbf{y} \leqslant 0$, define $\mathbf{F}(\mathbf{x}; \mathbf{y})$ to be the time of the first move involving the point $(\mathbf{x}; \mathbf{y})$, and $\mathbf{L}(\mathbf{x}; \mathbf{y})$ to be the time of the last one. (These must be well defined: every point in the starting army must be involved in at least one move, and since no point may be involved in infinitely many moves there cannot be any problem in identifying a well-defined first or last move.)

We consider the points on the negative $\mathbf{y}$-axis only: we will show that for all $0 \geqslant \mathbf{y}_0 > \mathbf{y}_1$, we have $\mathbf{F}(0; \mathbf{y}_0) \leqslant \mathbf{F}(0; \mathbf{y}_1)$ but $\mathbf{L}(0; \mathbf{y}_0) \geqslant \mathbf{L}(0; \mathbf{y}_1)$.

Let $(0; \mathbf{y}_0)$ be a point on the $\mathbf{y}$-axis, with $\mathbf{y}_0 \leqslant 0$, and let $\mathbf{S} = \{(0; \mathbf{y}) : \mathbf{y} < \mathbf{y}_0\}$ be the subset of the board consisting of the points directly below it. Let $\mathbf{T}_0$ be the time interval $\left[\mathbf{t}_{\text{start}}; \mathbf{F}(0; \mathbf{y}_0)\right)$, and let $\mathbf{T}_1$ be the time interval $\left(\mathbf{L}(0; \mathbf{y}_0); \mathbf{t}_{\text{end}}\right]$. Define

$$\mathbf{V}_S(\mathbf{t}) = \sum_{(x,y) \in S} \mathbf{f}(\mathbf{x}; \mathbf{y}; \mathbf{t}) \, \varphi^{\,y-|x|}$$

to be the total board value of the pegs within $\mathbf{S}$ at time $\mathbf{t}$.

During either of these time intervals, no move can take place which decreases $\mathbf{V}_S(\mathbf{t})$. A sideways move into $\mathbf{S}$ would increase $\mathbf{V}_S(\mathbf{t})$, while an upward move contained within $\mathbf{S}$ would leave $\mathbf{V}_S(\mathbf{t})$ unchanged, so the only way to make a move decreasing $\mathbf{V}_S(\mathbf{t})$ would be to move a peg upwards out of $\mathbf{S}$. But such a move would have to involve the point $(0; \mathbf{y}_0)$, and hence cannot by definition occur before $\mathbf{F}(0; \mathbf{y}_0)$ or after $\mathbf{L}(0; \mathbf{y}_0)$.

By defining a value function which matches our normal one on $\mathbf{S}$ and is zero everywhere else, we can apply Corollary 1 to show that if no individual move decreases $\mathbf{V}_S(\mathbf{t})$ during a given time interval then the overall value of $\mathbf{V}_S(\mathbf{t})$ cannot decrease during that interval. Hence, $\mathbf{V}_S(\mathbf{t})$ must be increasing on each of the intervals $\mathbf{T}_0$ and $\mathbf{T}_1$.

At the beginning of $\mathbf{T}_0$, $\mathbf{S}$ is completely full of pegs. Hence, it must remain completely full throughout $\mathbf{T}_0$, since any other configuration would yield a smaller value for $\mathbf{V}_S(\mathbf{t})$, violating our deduction that $\mathbf{V}_S(\mathbf{t})$ cannot decrease during $\mathbf{T}_0$. Similarly, at the end of $\mathbf{T}_1$, $\mathbf{S}$ is completely empty of pegs, and hence it must have remained completely empty throughout $\mathbf{T}_1$.

Hence, no point within $\mathbf{S}$ can have its first move within the interval $\mathbf{T}_0$, or its last move within $\mathbf{T}_1$. So given any $\mathbf{y}_1 < \mathbf{y}_0$, $\mathbf{F}(0; \mathbf{y}_1)$ must be at or after $\mathbf{F}(0; \mathbf{y}_0)$, and $\mathbf{L}(0; \mathbf{y}_1)$ must be at or before $\mathbf{L}(0; \mathbf{y}_0)$.

So we have shown, as desired, that for all $0 \geqslant \mathbf{y}_0 > \mathbf{y}_1$, we have $\mathbf{F}(0; \mathbf{y}_0) \leqslant \mathbf{F}(0; \mathbf{y}_1)$ and $\mathbf{L}(0; \mathbf{y}_0) \geqslant \mathbf{L}(0; \mathbf{y}_1)$. Equality can only occur when $|\mathbf{y}_0 - \mathbf{y}_1| < 3$, by the one-move-at-a-time rule; so we must have $F(0,0) < F(0,-3) < F(0,-6) < \dots$ being an infinite ascending sequence of move times, and $L(0,0) > L(0,-3) > L(0,-6) > \dots$ being an infinite descending sequence. The set formed by the elements of the former sequence has no greatest element, and the set formed by the latter has no least element; hence the set of move times cannot be well-ordered either forwards or backwards. □

This demonstrates that in permitting both forward and backward infinite move sequences, we have altered the rules of the game by *only just* enough to render a solution possible.

7. n dimensions

We now consider the generalisation of the Solitaire Army problem to $\mathbb{Z}^n$. Suppose our starting army is $\{(x_1, \dots, x_n) : x_n \leqslant 0\}$, moves can be made parallel to any coordinate axis, and our aim is to get as high up the x_n-axis as possible.

The obvious generalisation of the board value function is to let the value of the board position $(x_1, \dots, x_n)$ equal $\varphi^{x_n - |x_1| - \dots - |x_{n-1}|}$. This has the usual property that moves inwards or upwards preserve the value and all other moves decrease it. Adding up the total value of the starting position gives φ^{3n-1}; we might therefore expect that one ought to be able to reach $(0, \dots, 0, 3n-2)$ with finitely many moves, and $(0, \dots, 0, 3n-1)$ with infinite move sequences.

A construction demonstrating the finitely-many-moves case is given in [2]. We now present one which reaches $x_n = 3n - 1$, using infinitely many moves, for all $n \geqslant 3$. (We are of course using the rules given in section 2, with the uniform move bound. Under the relaxed rules, the problem becomes trivial exactly as it did in $\mathbb{Z}^2$.)

We begin by presenting an n-dimensional generalisation of the megawhoosh. Our "n-whoosh" will convert the region $\{(x_1, \dots, x_n) : x_1, \dots, x_{n-1} > 0, x_n \leqslant 0\}$ into a single peg at $(0, \dots, 0, n+1)$.

We define our n-whoosh inductively. Our base case is the 2-dimensional megawhoosh defined in section 3.2. Now suppose we have an $(n-1)$-whoosh and wish to construct an n-whoosh.

Working in reverse as usual, we begin with a single peg at $(0, \dots, 0, n+1)$. We start by moving it in the positive x_1 direction, so we have pegs at $(1, 0, \dots, 0, n+1)$ and $(2, 0, \dots, 0, n+1)$.

We now perform an $(n-1)$-whoosh in each of the hyperplanes $x_1 = 1$ and $x_1 = 2$. Each of these starts one space higher than it would normally do, so the result is a set of pegs occupying the positions $\{(x_1, \dots, x_n) : x_1 \in \{1, 2\}, x_2, \dots, x_{n-1} > 0, x_n \leqslant 1\}$.

Consider the cross-section of this set obtained by fixing constant positive integer values for

$x_2, \ldots, x_{n-1}$. The resulting plane (indexed by coordinates (x_1, x_n)) contains pegs in exactly the two columns $\{x_1 \in \{1, 2\}, x_n \leqslant 1\}$. This is the same as the position reached during our original megawhoosh construction, after the initial move and pair of whooshes. Hence, we can now perform the rest of the megawhoosh construction in this plane, and end up with the set of pegs $\{x_1 > 0, x_n \leqslant 0\}$. We repeat this for all the other similar cross-sections (of which there are countably many, so we can do them all in finite time), and we end up with the set $\{(x_1, \ldots, x_n) : x_1, \ldots, x_{n-1} > 0, x_n \leqslant 0\}$ as desired.

Now consider the complete starting position in $\mathbb{Z}^n$. It can be divided up into

- the central column $\{x_1 = \ldots = x_{n-1} = 0\,;\, x_n \leqslant 0\}$ which can be whooshed up to $(0, \ldots, 0, 2)$
- some quarter-planes such as $\{x_1 = \ldots = x_{n-2} = 0\,;\, x_{n-1} > 0\,;\, x_n \leqslant 0\}$ which can be megawhooshed up to $(0, \ldots, 0, 3)$
- some three-dimensional 'corners' such as $\{x_1 = \ldots = x_{n-3} = 0\,;\, x_{n-2}, x_{n-1} > 0\,;\, x_n \leqslant 0\}$ which can be 3-whooshed up to $(0, \ldots, 0, 4)$

  ...
- some n-dimensional segments such as $\{x_1, \ldots, x_{n-1} > 0\,;\, x_n \leqslant 0\}$ which can be n-whooshed up to $(0, \ldots, 0, n+1)$.

Suppose for a moment that we had an inexhaustible supply of $(n-1)$-segments and n-segments, and hence we were able to generate as many pegs at $(0, \ldots, 0, n)$ and $(0, \ldots, 0, n+1)$ as we liked. We could then reach as high as we wanted on the x_n-axis by the following procedure:

1. n-whoosh an n-segment to height $n+1$.
2. $(n-1)$-whoosh an $(n-1)$-segment to n, and move the resulting peg upwards to $n+2$.
3. Repeat step 1. Move the peg at $n+1$ upwards to $n+3$.
4. Repeat steps 1–2. Move the peg at $n+2$ upwards to $n+4$.
5. Repeat steps 1–3. Move the peg at $n+3$ upwards to $n+5$.
6. Repeat steps 1–4. Move the peg at $n+4$ upwards to $n+6$.

   ...

k. Repeat steps 1–$(k-2)$. Move the peg at $n+(k-2)$ upwards to $n+k$.

By inspection we can see that the number of $\mathbf{n}$-segments and $(\mathbf{n}-1)$-segments required by this procedure are given by Fibonacci numbers. Specifically, in order to reach height $\mathbf{n}+\mathbf{k}$, we would use $\mathbf{F}_{k-1}$ $(\mathbf{n}-1)$-segments and $\mathbf{F}_k$ $\mathbf{n}$-segments. Hence, to reach height $3\mathbf{n}-1$, we set $\mathbf{k}=2\mathbf{n}-1$ and find that we need $\mathbf{F}_{2n-2}$ $(\mathbf{n}-1)$-segments and $\mathbf{F}_{2n-1}$ $\mathbf{n}$-segments.

However, we do not have that many large segments, and we do have a number of smaller segments we need to use up. We deal with this by combining smaller segments together: we can simulate the effect of $\mathbf{k}$-whooshing a $\mathbf{k}$-segment to height $\mathbf{k}+1$ by instead $(\mathbf{k}-1)$-whooshing a $(\mathbf{k}-1)$-segment to height $\mathbf{k}$, $(\mathbf{k}-2)$-whooshing a $(\mathbf{k}-2)$-segment to $\mathbf{k}-1$, and jumping the latter over the former. Segments combined in this way can be recombined: if we have some collection of segments which together generate a peg at height $\mathbf{j}$, and some other collection generating a peg at $\mathbf{j}-1$, then by combining the former collection first we can arrange for it to be out of the way of the second, and hence we can combine them to produce a peg at $\mathbf{j}+1$.

It remains to show that the set of segments we actually have available can be combined into collections which generate the right number of pegs at $\mathbf{n}-1$ and $\mathbf{n}$.

In an actual $\mathbf{n}$-dimensional starting position, we have segments of all dimensions from 1 (the central column) to $\mathbf{n}$. We can count the number of $\mathbf{k}$-segments, for any $\mathbf{k}$, by observing that a $\mathbf{k}$-segment is characterised by choosing $\mathbf{k}-1$ of the $\mathbf{n}-1$ coordinates to be non-zero, and then for each of those $\mathbf{k}-1$ coordinates, choosing whether that coordinate should be positive or negative. Hence, there are $2^{k-1}\binom{n-1}{k-1}$ $\mathbf{k}$-segments in total.

We transform this collection of segments by combining all of the smallest size of segment with the next size up, and then repeating. So if we started off with $\mathbf{p}_k$ $\mathbf{k}$-segments, then we would transform them as follows:

$$
\begin{array}{ll}
 & (\mathbf{p}_1; \mathbf{p}_2; \mathbf{p}_3; \ldots; \mathbf{p}_n) \\
\to & (0; \mathbf{p}_2-\mathbf{p}_1; \mathbf{p}_3+\mathbf{p}_1; \mathbf{p}_4; \ldots; \mathbf{p}_n) \\
\to & (0; 0; \mathbf{p}_3-\mathbf{p}_2+2\mathbf{p}_1; \mathbf{p}_4+\mathbf{p}_2-\mathbf{p}_1; \mathbf{p}_5; \ldots; \mathbf{p}_n) \\
\to & (0; 0; 0; \mathbf{p}_4-\mathbf{p}_3+2\mathbf{p}_2-3\mathbf{p}_1; \mathbf{p}_5+\mathbf{p}_3-\mathbf{p}_2+2\mathbf{p}_1; \mathbf{p}_6; \ldots; \mathbf{p}_n) \\
\to & (0; 0; 0; 0; \mathbf{p}_5-\mathbf{p}_4+2\mathbf{p}_3-3\mathbf{p}_2+5\mathbf{p}_1; \mathbf{p}_6+\mathbf{p}_4-\mathbf{p}_3+2\mathbf{p}_2-3\mathbf{p}_1; \mathbf{p}_7; \ldots; \mathbf{p}_n) \\
\to & (0; 0; 0; 0; 0; \mathbf{p}_6-\mathbf{p}_5+2\mathbf{p}_4-3\mathbf{p}_3+5\mathbf{p}_2-8\mathbf{p}_1; \mathbf{p}_7+\mathbf{p}_5-\mathbf{p}_4+2\mathbf{p}_3-3\mathbf{p}_2+5\mathbf{p}_1; \mathbf{p}_8; \ldots; \mathbf{p}_n) \\
 & \vdots \\
\to & (0; \ldots; 0; \mathbf{p}'_{n-1}; \mathbf{p}'_n)
\end{array}
$$

where, by inspection,

$$
\begin{aligned}
\mathbf{p}'_{n-1} &= \mathbf{F}_1\mathbf{p}_{n-1}-\mathbf{F}_2\mathbf{p}_{n-2}+\mathbf{F}_3\mathbf{p}_{n-3}-\ldots+(-1)^n\mathbf{F}_{n-1}\mathbf{p}_1 \\
\mathbf{p}'_n &= \mathbf{p}_n+\mathbf{F}_1\mathbf{p}_{n-2}-\mathbf{F}_2\mathbf{p}_{n-3}+\mathbf{F}_3\mathbf{p}_{n-4}-\ldots+(-1)^{n-1}\mathbf{F}_{n-2}\mathbf{p}_1 \\
&= \mathbf{F}_{-1}\mathbf{p}_n-\mathbf{F}_0\mathbf{p}_{n-1}+\mathbf{F}_1\mathbf{p}_{n-2}-\mathbf{F}_2\mathbf{p}_{n-3}+\mathbf{F}_3\mathbf{p}_{n-4}-\ldots+(-1)^{n-1}\mathbf{F}_{n-2}\mathbf{p}_1
\end{aligned}
$$

(extending the definition of Fibonacci numbers backwards from F_1 in the obvious way to get $\mathsf{F}_{-1} = 1$ and $\mathsf{F}_0 = 0$, which simplifies the notation).

We can write these formulae as follows:

$$\begin{aligned} \mathsf{p}'_{n-1} &= \sum_{k=1}^{n-1} (-1)^{k-1} \mathsf{F}_k \mathsf{p}_{n-k} \\ \mathsf{p}'_n &= \sum_{k=-1}^{n-2} (-1)^{k-1} \mathsf{F}_k \mathsf{p}_{n-1-k} \end{aligned}$$

If we then substitute in $\mathsf{p}_k = 2^{k-1}\binom{n-1}{k-1}$ and expand using Binet's formula for F_n, we find that $\mathsf{p}'_{n-1} = \mathsf{F}_{2n-2}$ and $\mathsf{p}'_n = \mathsf{F}_{2n-1}$ (this is entirely mechanical and left to the reader). Hence we have F_{2n-2} combinations of segments which are each functionally equivalent to a $(\mathbf{n}-1)$-segment, and F_{2n-1} combinations each functionally equivalent to an $\mathbf{n}$-segment. This is exactly the number we deduced earlier that we need to reach the target position.

Web page

We have a web page containing an animated version of the solution at

`http://www.chiark.greenend.org.uk/~sgtatham/solarmy/`

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