PROBLEM (*Ponder This*, June 2011).

A circular road is divided into 100 sectors. One by one, cars park across a pair of adjacent unoccupied sectors, chosen uniformly at random among all such pairs. Eventually, no such pairs remain, so no more cars can park. On average, how many cars find a space?

SOLUTION (Austin Shapiro).

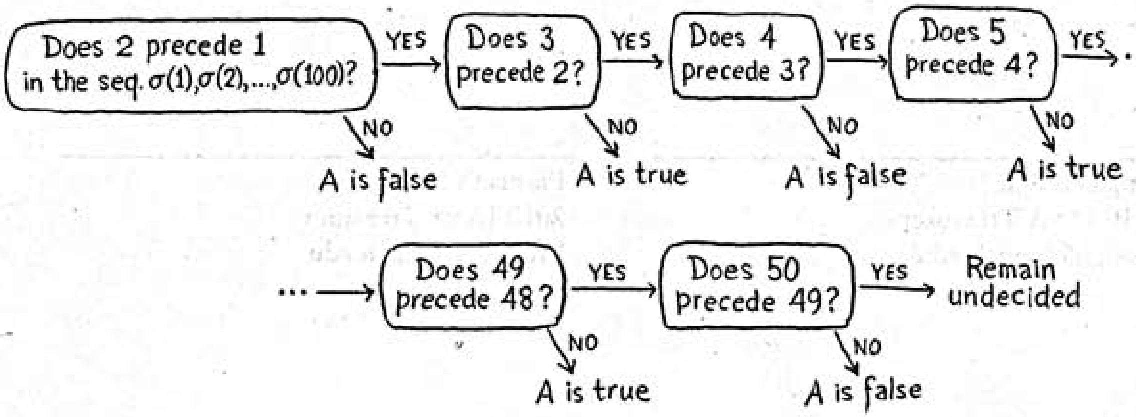
Let us label the sectors $1, 2, \ldots, 100$ (modulo 100, so that $100+1=1$). Most of our effort will go into computing the probability that a given sector — say, sector 1 — is never occupied.

If σ is a random permutation of $\{1,2,\ldots,100\}$, we can model the random selection of parking spaces by having the i^{th} car attempt to park at $\{\sigma(i), \sigma(i)+1\}$; if that space is partially or wholly occupied, we have the i^{th} car drive away. (The cars that drive away are immaterial to the problem!)

Now let event A be *'A car successfully parks at $\{2,3\}$ before a car attempts to park at $\{1,2\}$.'*
Let event B be *'A car successfully parks at $\{99,100\}$ before a car attempts to park at $\{100,1\}$.'*

Then sector 1 is never occupied if and only if events A and B both hold.

We can use this decision tree to evaluate the truth or falsehood of A:



The probability of ever reaching the i^{th} node in the decision tree is $\frac{1}{i!}$, as this occurs if and only if σ reverses the order of $1, 2, \ldots, i$ completely.

Thus, the decision procedure returns 'A is true' with probability $\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{4!}-\frac{1}{5!}\right)+\cdots+\left(\frac{1}{48!}-\frac{1}{49!}\right)=\frac{D_{49}}{49!}$ (where D_n denotes the number of derangements of n). More precisely, when the tree returns 'A is true', we have satisfied sufficient conditions for A which depend only on the ordering induced by σ on $\{1,\ldots,50\}$. When the tree fails to return 'A is false' (which occurs with probability $\frac{D_{50}}{50!}$), we have satisfied necessary conditions for A which depend only on the ordering of $\{1,\ldots,50\}$.

By symmetry, we can build a decision tree for B which verifies sufficient (resp., necessary) conditions for B with probability $\frac{D_{49}}{49!}$ (resp., $\frac{D_{50}}{50!}$), such that these conditions depend only on the ordering induced by σ on $\{51,\ldots,100\}$. Since the orderings on $\{1,\ldots,50\}$ and $\{51,\ldots,100\}$ are independent, we infer

$$\left(\frac{D_{49}}{49!}\right)^2 \le \Pr(A \text{ and } B) \le \left(\frac{D_{50}}{50!}\right)^2.$$

These bounds are both equal to $\frac{1}{e^2}$, up to a tolerance of $\frac{2}{e}\cdot\frac{1}{50!}$.

The average number of cars is therefore $\frac{1}{2}\left[100\left(1-\frac{1}{e^2}\right)\right] \approx 43.2332358$, to within an extremely small error.